Topic 8 -
Undetermined Coefficients

Topic 8 - Method of undetermined coefficients

We want to be able to solve:

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

Where a_2, a_1, a_0 are constants.

Method:

① Find general solution y_h to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

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② Guess a particular solution y_p to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

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③ The general solution to

use topic

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

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is

$$y = y_h + y_p$$

This table is on handout on the website. For guessing y_p.

$b(x)$	y_p guess for undetermined coefficients
constant	A
degree one polynomial such as: $5x - 3$ or $2x$	$Ax + B$
degree two polynomial such as: $10x^2 - x + 1$ or $x^2 + x$ or $2x^2 - 3$	$Ax^2 + Bx + C$
$\sin(kx)$ where k is a constant	$A\cos(kx) + B\sin(kx)$
$\cos(kx)$ where k is a constant	$A\cos(kx) + B\sin(kx)$
exponential such as: e^{kx} or $-2e^{kx}$	Ae^{kx}
degree one poly times exponential such as: xe^{kx} or $(2x+1)e^{kx}$	$(Ax + B)e^{kx}$

Ex: Find the general solution to

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

Factor way:

$$(r+1)(r+2) = 0$$

$$r = -1, -2$$

The roots are:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2} = \boxed{-1, -2}$$

We get

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2: Guess a solution y_p to

$$y'' + 3y' + 2y = 2x^2$$

Let's guess

$$y_p = Ax^2 + Bx + C$$

← A, B, C unknown numbers

We have

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug these into the equation to get

$$\underbrace{(2A)}_{y_p''} + 3\underbrace{(2Ax+B)}_{y_p'} + 2\underbrace{(Ax^2+Bx+C)}_{y_p} = 2x^2$$

We get

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

Regrouping:

$$\underbrace{\boxed{2A}}_{2}x^2 + \underbrace{\boxed{(6A+2B)}}_{0}x + \underbrace{\boxed{(2A+3B+2C)}}_{0} = \underbrace{2x^2}_{2x^2+0x+0}$$

Get:

$$2A = 2 \quad (1)$$
$$6A + 2B = 0 \quad (2)$$
$$2A + 3B + 2C = 0 \quad (3)$$

(1) gives $A = 1$.

Plug into (2) and get $6(1) + 2B = 0$.

$$B = -3$$

Plug into (3) and get $2(1) + 3(-3) + 2C = 0$

$$C = 7/2$$

Thus,

$$y_p = Ax^2 + Bx + C = x^2 - 3x + \frac{7}{2}$$

Step 3: The general solution to

$$y'' + 3y' + 2y = 2x^2$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + \frac{7}{2}}$$

Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Solve

$$y'' - y' + y = 0$$

The characteristic equation is

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{1 \pm \sqrt{3}\, i}{2} = \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\alpha \pm \beta i}$$

$$\alpha = 1/2, \ \beta = \sqrt{3}/2$$

General formula:

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

In our case we get:

$$y_h = c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Step 2: Guess a solution to

$$y'' - y' + y = \underbrace{2\sin(3x)}_{b(x)}$$

Guess:

$$y_p = A\cos(3x) + B\sin(3x)$$

We get

$$y_p' = -3A\sin(3x) + 3B\cos(3x)$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x)$$

Plug these into the equation:

$$\left.\begin{aligned}&(-9A\cos(3x) - 9B\sin(3x))\\-&(-3A\sin(3x) + 3B\cos(3x))\\+&(A\cos(3x) + B\sin(3x))\end{aligned}\right]\begin{aligned}&y_p''\\-&y_p'\\+&y_p\end{aligned}$$

$$= 2\sin(3x)$$

Regrouping:

$$\underbrace{(3A - 8B)}_{2}\sin(3x) + \underbrace{(-8A - 3B)}_{0}\cos(3x)$$

$$= 2\sin(3x)$$

Need

$$\boxed{\begin{aligned}3A - 8B &= 2\\-8A - 3B &= 0\end{aligned}}\quad\begin{aligned}&①\\&②\end{aligned}$$

② gives $A = -\frac{3}{8}B$.

Plug into ①: $3\underbrace{\left(-\frac{3}{8}B\right)}_{A} - 8B = 2$

$$-\frac{9}{8}B - 8B = 2$$

$$-\frac{73}{8}B = 2$$

$$\boxed{B = \frac{-16}{73}}$$

So, $A = \frac{-3}{8}B = \frac{-3}{8}\left(\frac{-16}{73}\right) = \boxed{\frac{6}{73}}$

Thus,

$$y_p = \underbrace{\frac{6}{73}}_{A}\cos(3x) - \underbrace{\frac{16}{73}}_{B}\sin(3x)$$

Step 3: The general solution to

$$y'' - y' + y = 2\sin(3x)$$

is

$$y = y_h + y_p$$

$$y = c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x)$$

What can go wrong with the guessing method for y_p?

If your y_p guess appears as a term in y_h then you need to multiply your guess by powers of x until your guess doesn't appear as a term in y_h

Ex: Solve

$$y'' - 5y' + 4y = 8e^x$$

Step 1: Solve

$$y'' - 5y' + 4y = 0$$

The characteristic equation is

$$r^2 - 5r + 4 = 0$$

$(r-4)(r-1) = 0$

$r = 4, 1$

The roots are

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= \frac{5+3}{2}, \frac{5-3}{2} = \boxed{4, 1}$$

The solution to $y'' - 5y' + 4y = 0$

is $y_h = c_1 e^{4x} + c_2 e^{x}$

Step 2: Guess a solution y_p to

$$y'' - 5y' + 4y = \underbrace{8e^{x}}_{b(x)}$$

The table says to guess

$$y_p = Ae^{x}$$

This won't work, $y_p = Ae^{x}$ appears as a term in y_h.

Try plugging it into

We have $y_p = Ae^{x}$, $y_p' = Ae^{x}$, $y_p'' = Ae^{x}$

Plugging into the equation gives:

$$\underbrace{(Ae^x) - 5(Ae^x) + 4(Ae^x)}_{y'' - 5y' + 4y} = 8e^x$$

This gives

$$0 = 8e^x$$

This isn't solvable.

Since $y_p = Ae^x$ appears as a term in $y_h = c_1 e^{4x} + c_2 e^x$ we need to multiply our guess by an x.

Instead guess: $\boxed{y_p = Axe^x}$

We have
$$y_p' = Ae^x + Axe^x$$
$$y_p'' = Ae^x + (Ae^x + Axe^x)$$
$$= 2Ae^x + Axe^x$$

Now plug into $y'' - 5y' + 4y = 8e^x$ to get:

$$\underbrace{(2Ae^x + Axe^x)}_{y_p''} - 5\underbrace{(Ae^x + Axe^x)}_{y_p'} + 4\underbrace{(Axe^x)}_{y_p} = 8e^x$$

This gives:

$$2Ae^x + Axe^x - 5Ae^x - 5Axe^x + 4Axe^x = 8e^x$$

($Axe^x - 5Axe^x + 4Axe^x = 0$)

We get:

$$-3Ae^x = 8e^x$$

Need

$$-3A = 8$$

So, $A = -8/3$

Thus, $y_p = \frac{-8}{3} x e^x$

solves $y'' - 5y' + 4y = 8e^x$

Step 3: The general solution to

$$y'' - 5y' + 4y = 8e^{x}$$

is

$$y = y_h + y_p$$
$$= c_1 e^{4x} + c_2 e^{x} - \frac{8}{3} x e^{x}$$

Ex: Solve

$$y'' - 2y' + y = e^{x}$$

Step 1: Solve

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$r^2 - 2r + 1 = 0$$

We get

$$(r-1)(r-1) = 0$$

So we get a repeated real root $r = 1$

Then,

$$y_h = c_1 e^x + c_2 x e^x$$

is the general solution to

$$y'' - 2y' + y = 0$$

Step 2: Now we guess y_p for

$$y'' - 2y' + y = \underbrace{e^x}_{b(x)}$$

The table says to guess $y_p = Ae^x$

But this appears in $y_h = c_1 e^x + c_2 x e^x$

So multiply our guess by x and guess $y_p = Axe^x$

But this appears in $y_h = c_1 e^x + c_2 x e^x$

So multiply our guess by x again to get $y_p = Ax^2e^x$ ← doesn't appear in y_h

Now we plug it in.

$$y_p = Ax^2e^x$$

$$y_p' = 2Axe^x + Ax^2e^x$$

$$y_p'' = (2Ae^x + 2Axe^x) + (2Axe^x + Ax^2e^x)$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x$$

Plug these into

$$y'' - 2y' + y = e^x$$

to get:

$$(2Ae^x + 4Axe^x + Ax^2e^x) - 2(2Axe^x + Ax^2e^x) + Ax^2e^x = e^x$$

This gives:

$$2Ae^x + 4Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x + Ax^2e^x = e^x$$

(the $4Axe^x$ terms sum to 0; the x^2e^x terms sum to 0)

We get:

$$2Ae^x = e^x$$

Need

$$2A = 1$$

So,

$$A = 1/2$$

Thus, $y_p = \frac{1}{2}x^2e^x$ solves

$$y'' - 2y' + y = e^x$$

Step 3: The general solution to

$$y'' - 2y' + y = e^x$$

is

$$y = y_h + y_p = \boxed{c_1 e^x + c_2 x e^x + \frac{1}{2} x^2 e^x}$$

There is also the situation where the $b(x)$ on the right side of the equation is a sum of terms. In this case your guess would be a sum of terms, one for each term in $b(x)$

Ex: For $y'' + 2y' = 2x + 5 - e^x$

you would guess:

$$y_p = Ax + B + Ce^x$$

Ex: For $y'' + y = 2\sin(x) + x^2$

you would guess:

$$y_p = A\sin(x) + B\cos(x) + Cx^2 + Dx + E$$